The maximum sum and maximum product of sizes of cross-intersecting families

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May 6, 2013

Abstract

We say that a set A *t-intersects* a set B if A and B have at least t common elements. A family $\mathcal{A}$ of sets is said to be *t-intersecting* if each set in $\mathcal{A}$ t-intersects any other set in $\mathcal{A}$. Families $\mathcal{A}_1, \mathcal{A}_2, ..., \mathcal{A}_k$ are said to be *cross-t-intersecting* if for any i and j in $\{1, 2, ..., k\}$ with $i \neq j$, any set in $\mathcal{A}_i$ t-intersects any set in $\mathcal{A}_j$. We prove that for any finite family $\mathcal{F}$ that has at least one set of size at least t, there exists an integer $\kappa \leq |\mathcal{F}|$ such that for any $k \geq \kappa$, both the sum and the product of sizes of any k cross-t-intersecting subfamilies $\mathcal{A}_1, ..., \mathcal{A}_k$ (not necessarily distinct or non-empty) of $\mathcal{F}$ are maxima if $\mathcal{A}_1 = ... = \mathcal{A}_k = \mathcal{L}$ for some largest t-intersecting subfamily $\mathcal{L}$ of $\mathcal{F}$. We then study the smallest possible value of κ and investigate the case $k < \kappa$; this includes a cross-intersection result for straight lines that demonstrates that it is possible to have $\mathcal{F}$ and κ such that for any $k < \kappa$, the configuration $\mathcal{A}_1 = ... = \mathcal{A}_k = \mathcal{L}$ is neither optimal for the sum nor optimal for the product. We also outline solutions for various important families $\mathcal{F}$, and we provide solutions for the case when $\mathcal{F}$ is a power set.

1 Introduction

Unless otherwise stated, throughout this paper we shall use small letters such as x to denote elements of a set or positive integers, capital letters such as X to denote sets, and calligraphic letters such as $\mathcal{F}$ to denote *families* (that is, sets whose elements are sets themselves). Unless specified, sets and families are taken to be finite and may be the *empty set* $\emptyset$. An *r-set* is a set of size r, that is, a set having exactly r elements. For any integer $n \geq 1$, $[n]$ denotes the set $\{1, ..., n\}$ of the first n positive integers.

Given an integer $t \geq 1$, we say that a set A *t-intersects* a set B if A and B have at least t common elements. A family $\mathcal{A}$ is said to be *t-intersecting* if each set in $\mathcal{A}$ t-intersects any other set in $\mathcal{A}$ (i.e. $|A \cap B| \geq t$ for any $A, B \in \mathcal{A}$ with $A \neq B$). A 1-intersecting family is also simply called an *intersecting family*. Families $\mathcal{A}_1, ..., \mathcal{A}_k$ are said to be *cross-t-intersecting* if for any i and j in $[k]$ with $i \neq j$, any set in $\mathcal{A}_i$ t-intersects any set in $\mathcal{A}_j$ (i.e. $|A \cap B| \geq t$ for any $A \in \mathcal{A}_i$ and any $B \in \mathcal{A}_j$). Cross-1-intersecting families are also simply called *cross-intersecting families*.

Let $\binom{[n]}{r}$ denote the family of all subsets of $[n]$ of size r. The classical Erdős-Ko-Rado (EKR) Theorem [17] says that if n is sufficiently larger than r, then the size of any t-intersecting subfamily of $\binom{[n]}{r}$ is at most $\binom{n-t}{r-t}$, which is the number of sets in the

t-intersecting subfamily of $\binom{[n]}{r}$ consisting of those sets having $[t]$ as a subset. The EKR Theorem inspired a wealth of results of this kind, that is, results that establish how large a system of sets can be under certain intersection conditions; see [10, 14, 18].

For t-intersecting subfamilies of a given family $\mathcal{F}$, the natural question to ask is how large they can be. For cross-t-intersecting families, two natural parameters arise: the sum and the product of sizes of the cross-t-intersecting families (note that the product of sizes of k families $\mathcal{A}_1, ..., \mathcal{A}_k$ is the number of k-tuples $(A_1, ..., A_k)$ such that $A_i \in \mathcal{A}_i$ for each $i \in [k]$). It is therefore natural to consider the problem of maximising the sum or the product of sizes of k cross-t-intersecting subfamilies (not necessarily distinct or non-empty) of a given family $\mathcal{F}$.

The main result in this paper (Theorem 1.1 below) relates both the maximum sum and the maximum product of sizes of k cross-t-intersecting subfamilies of any family $\mathcal{F}$ to the maximum size of a t-intersecting subfamily of $\mathcal{F}$ when k is not smaller than a certain value depending on $\mathcal{F}$ and t. It gives the maximum sum and the maximum product in terms of the size of a largest t-intersecting subfamily.

For any non-empty family $\mathcal{F}$, let $\alpha(\mathcal{F})$ denote the size of a largest set in $\mathcal{F}$. Suppose $\alpha(\mathcal{F}) < t$, and let $\mathcal{A}_1, ..., \mathcal{A}_k$ ($k \geq 2$) be subfamilies of $\mathcal{F}$. Then $\mathcal{A}_1, ..., \mathcal{A}_k$ are cross-t-intersecting if and only if at most one of them is non-empty (since no set in $\mathcal{F}$ t-intersects itself or another set in $\mathcal{F}$). Thus, if $\mathcal{A}_1, ..., \mathcal{A}_k$ are cross-t-intersecting, then the product of their sizes is 0 and the sum of their sizes is at most the size $|\mathcal{F}|$ of $\mathcal{F}$ (which is attained if and only if one of them is $\mathcal{F}$ and the others are all empty). This completely solves our problem for the case $\alpha(\mathcal{F}) < t$.

We now address the case $\alpha(\mathcal{F}) \geq t$. Before stating our main result, we need to introduce some definitions and parameters.

For any family $\mathcal{A}$, let $\mathcal{A}^{t,+}$ be the (t-intersecting) subfamily of $\mathcal{A}$ given by

$$\mathcal{A}^{t,+} = \{A \in \mathcal{A} \colon |A \cap B| \geq t \text{ for any } B \in \mathcal{A} \text{ such that } A \neq B\},$$

and let

$$\mathcal{A}^{t,-} = \mathcal{A} \backslash \mathcal{A}^{t,+}.$$

In simple terms, a set A in $\mathcal{A}$ is in $\mathcal{A}^{t,-}$ if there exists a set B in $\mathcal{A}$ such that $A \neq B$ and A does not t-intersect B, otherwise A is in $\mathcal{A}^{t,+}$. The definitions of $\mathcal{A}^{t,+}$ and $\mathcal{A}^{t,-}$ are generalisations of the definitions of $\mathcal{A}^*$ and $\mathcal{A}'$ in [5, 6, 7, 8, 12]; $\mathcal{A}^* = \mathcal{A}^{1,+}$ and $\mathcal{A}' = \mathcal{A}^{1,-}$.

Let $l(\mathcal{F}, t)$ denote the size of a largest t-intersecting subfamily of a non-empty family $\mathcal{F}$. For any subfamily $\mathcal{A}$ of $\mathcal{F}$, we define

$$\beta(\mathcal{F}, t, \mathcal{A}) = \begin{cases} \dfrac{l(\mathcal{F}, t) - |\mathcal{A}^{t,+}|}{|\mathcal{A}^{t,-}|} & \text{if } \mathcal{A}^{t,-} \neq \emptyset; \\ \dfrac{l(\mathcal{F}, t)}{|\mathcal{F}|} & \text{if } \mathcal{A}^{t,-} = \emptyset; \end{cases}$$

so $|\mathcal{A}^{t,+}| + \beta(\mathcal{F}, t, \mathcal{A})|\mathcal{A}^{t,-}| \leq l(\mathcal{F}, t)$ (even if $\mathcal{A}^{t,-} = \emptyset$, because $|\mathcal{A}^{t,+}| \leq l(\mathcal{F}, t)$ since $\mathcal{A}^{t,+}$ is t-intersecting). We now define

$$\beta(\mathcal{F}, t) = \min\{\beta(\mathcal{F}, t, \mathcal{A}) \colon \mathcal{A} \subseteq \mathcal{F}\}.$$

Therefore,

$$|\mathcal{A}^{t,+}| + \beta(\mathcal{F}, t)|\mathcal{A}^{t,-}| \leq l(\mathcal{F}, t) \quad \text{for any } \mathcal{A} \subseteq \mathcal{F}. \tag{1}$$

In Section 3 we show that in fact

$$\beta(\mathcal{F}, t) = \max\left\{c \in \mathbb{R} \colon c \leq \frac{l(\mathcal{F}, t)}{|\mathcal{F}|},\ |\mathcal{A}^{t,+}| + c|\mathcal{A}^{t,-}| \leq l(\mathcal{F}, t) \text{ for any } \mathcal{A} \subseteq \mathcal{F}\right\} \tag{2}$$

(see Proposition 3.2), where $\mathbb{R}$ is the set of real numbers, and we also determine other basic facts about the parameter $\beta(\mathcal{F}, t)$; in particular, we show that we actually have

$$\frac{1}{|\mathcal{F}|} \leq \beta(\mathcal{F}, t) \leq \frac{l(\mathcal{F}, t)}{|\mathcal{F}|} \tag{3}$$

(see Propositions 3.1 and 3.2). In Section 3.2 we point out various important families $\mathcal{F}$ for which $\beta(\mathcal{F}, t)$ is known to be $\frac{l(\mathcal{F},t)}{|\mathcal{F}|}$.

By the lower bound in (3), for any non-empty family $\mathcal{F}$, we can define

$$\kappa(\mathcal{F}, t) = \frac{1}{\beta(\mathcal{F}, t)}$$

and we have

$$\kappa(\mathcal{F}, t) \leq |\mathcal{F}|.$$

We can now state our main result.

Theorem 1.1 *Let $\mathcal{A}_1, ..., \mathcal{A}_k$ be cross-t-intersecting subfamilies of a family $\mathcal{F}$ with $\alpha(\mathcal{F}) \geq t$. If $k \geq \kappa(\mathcal{F}, t)$, then*

$$\sum_{i=1}^{k} |\mathcal{A}_i| \leq k(l(\mathcal{F}, t)) \quad \text{and} \quad \prod_{i=1}^{k} |\mathcal{A}_i| \leq (l(\mathcal{F}, t))^k,$$

and both bounds are attained if $\mathcal{A}_1 = ... = \mathcal{A}_k = \mathcal{L}$ for some largest t-intersecting subfamily $\mathcal{L}$ of $\mathcal{F}$. Moreover, if $k > \kappa(\mathcal{F}, t)$, then in both inequalities, equality holds only if $\mathcal{A}_1 = ... = \mathcal{A}_k = \mathcal{L}$ for some largest t-intersecting subfamily $\mathcal{L}$ of $\mathcal{F}$.

In Section 4 we prove the following result, which tells us that if $k < \kappa(\mathcal{F}, t)$, then the sum inequality above does not hold for $\mathcal{A}_1, ..., \mathcal{A}_k$ with a maximum value of $\sum_{i=1}^{k} |\mathcal{A}_i|$.

Theorem 1.2 *Let $\mathcal{F}$ be a family with $\alpha(\mathcal{F}) \geq t$. Let $\mathcal{A}_1, ..., \mathcal{A}_k$ be cross-t-intersecting subfamilies of $\mathcal{F}$ such that $\sum_{i=1}^{k} |\mathcal{A}_i|$ is maximum. Then:*
(i) $\sum_{i=1}^{k} |\mathcal{A}_i| = k(l(\mathcal{F}, t))$ if $k \geq \kappa(\mathcal{F}, t)$;
(ii) $\sum_{i=1}^{k} |\mathcal{A}_i| > k(l(\mathcal{F}, t))$ if $k < \kappa(\mathcal{F}, t)$.

In Theorem 1.1 the product inequality follows immediately from the sum inequality and the following elementary result, known as the Arithmetic Mean-Geometric Mean (AM-GM) Inequality.

Lemma 1.3 (AM-GM Inequality) *If $x_1, x_2, \ldots, x_k$ are non-negative real numbers, then*

$$\left(\prod_{i=1}^{k} x_i\right)^{1/k} \leq \frac{1}{k} \sum_{i=1}^{k} x_i.$$

Indeed, if $\sum_{i=1}^{k} |\mathcal{A}_i| \leq k(l(\mathcal{F}, t))$, then, by Lemma 1.3, $\left(\prod_{i=1}^{k} |\mathcal{A}_i|\right)^{1/k} \leq l(\mathcal{F}, t)$ and hence $\prod_{i=1}^{k} |\mathcal{A}_i| \leq (l(\mathcal{F}, t))^k$. Therefore, if the configuration $\mathcal{A}_1 = ... = \mathcal{A}_k = \mathcal{L}$ (where $\mathcal{L}$ is as in Theorem 1.1) gives a maximum sum, then $\mathcal{A}_1 = ... = \mathcal{A}_k = \mathcal{L}$ also gives a maximum product. The converse is not true; indeed, as demonstrated in Section 5, we may have that $2 \leq k < \kappa(\mathcal{F}, t)$ and $\mathcal{A}_1 = ... = \mathcal{A}_k = \mathcal{L}$ still gives a maximum product, in which case $\mathcal{A}_1 = ... = \mathcal{A}_h = \mathcal{L}$ gives a maximum product for any $h \geq k$ (see Lemma 5.1). However, in Section 5 we prove the following interesting fact.

Remark 1.4 Just like the threshold $\kappa(\mathcal{F}, t)$ for the maximum sum part of Theorem 1.1 cannot be improved (by Theorem 1.2), the threshold $\kappa(\mathcal{F}, t)$ for the maximum product part of Theorem 1.1 can neither be improved in general. Indeed, we will give a (geometrical) construction of a family $\mathcal{F}$ such that for any $k < \kappa(\mathcal{F}, t)$, the product of sizes of k cross-t-intersecting subfamilies $\mathcal{A}_1, ..., \mathcal{A}_k$ of $\mathcal{F}$ is not maximum if $\mathcal{A}_1 = ... = \mathcal{A}_k = \mathcal{L}$; see Construction 5.3 and Theorem 5.4.

We conclude this section by mentioning that in Sections 4 and 5 we provide various general results about the maximum sum and the maximum product, respectively, and we also outline solutions for various important families.

2 Proof of the main result

The proof of Theorem 1.1 relies on the following lemma.

Lemma 2.1 *Let $\mathcal{A}_1, ..., \mathcal{A}_k$ be cross-t-intersecting families, and let $\mathcal{A} = \bigcup_{i=1}^k \mathcal{A}_i$. Then*
(i) $\mathcal{A}^{t,+} = \bigcup_{i=1}^k {\mathcal{A}_i}^{t,+}$,
(ii) $\mathcal{A}^{t,-} = \bigcup_{i=1}^k {\mathcal{A}_i}^{t,-}$,
(iii) $|\mathcal{A}^{t,-}| = \sum_{i=1}^k |{\mathcal{A}_i}^{t,-}|$.

Proof. Clearly $\mathcal{A}^{t,+} \subseteq \bigcup_{i=1}^k {\mathcal{A}_i}^{t,+}$. Suppose $A \in \bigcup_{i=1}^k {\mathcal{A}_i}^{t,+}$. Then $A \in {\mathcal{A}_h}^{t,+}$ for some $h \in [k]$, meaning that $|A \cap B| \geq t$ for any $B \in \mathcal{A}_h \backslash \{A\}$. Also, by the cross-$t$-intersection condition, for any $j \in [k] \backslash \{h\}$, $|A \cap B| \geq t$ for any $B \in \mathcal{A}_j$. So $A \in \mathcal{A}^{t,+}$. Therefore, $\bigcup_{i=1}^k {\mathcal{A}_i}^{t,+} \subseteq \mathcal{A}^{t,+}$. Together with $\mathcal{A}^{t,+} \subseteq \bigcup_{i=1}^k {\mathcal{A}_i}^{t,+}$, this gives us (i).

Clearly $\bigcup_{i=1}^k {\mathcal{A}_i}^{t,-} \subseteq \mathcal{A}^{t,-}$. Suppose $A \in \mathcal{A}^{t,-}$. Then $A \in \mathcal{A}_h$ for some $h \in [k]$, and $|A \cap B| < t$ for some $B \in \mathcal{A} \backslash \{A\}$. By the cross-$t$-intersection condition, $B \notin \mathcal{A}_j$ for each $j \in [k] \backslash \{h\}$. So $B \in \mathcal{A}_h$ and hence $A \in {\mathcal{A}_h}^{t,-}$. Therefore, $\mathcal{A}^{t,-} \subseteq \bigcup_{i=1}^k {\mathcal{A}_i}^{t,-}$. Together with $\bigcup_{i=1}^k {\mathcal{A}_i}^{t,-} \subseteq \mathcal{A}^{t,-}$, this gives us (ii).

Suppose ${\mathcal{A}_i}^{t,-} \cap {\mathcal{A}_j}^{t,-} \neq \emptyset$ for some i and j in $[k]$ with $i \neq j$. Let $A \in {\mathcal{A}_i}^{t,-} \cap {\mathcal{A}_j}^{t,-}$. Having $A \in {\mathcal{A}_i}^{t,-}$ means that there exists a set B in ${\mathcal{A}_i}^{t,-}$ such that $|A \cap B| < t$; however, since $A \in \mathcal{A}_j$, this contradicts the cross-t-intersection condition. So ${\mathcal{A}_i}^{t,-} \cap {\mathcal{A}_j}^{t,-} = \emptyset$ for any i and j in $[k]$ with $i \neq j$ (that is, ${\mathcal{A}_1}^{t,-}, ..., {\mathcal{A}_k}^{t,-}$ are disjoint). Together with (ii), this gives us (iii). $\Box$

Proof of Theorem 1.1. Suppose $k \geq \kappa(\mathcal{F}, t)$. Then $\beta(\mathcal{F}, t) \geq 1/k$. Let $\mathcal{A}$ be the subfamily of $\mathcal{F}$ given by the union $\bigcup_{i=1}^k \mathcal{A}_i$. We have

$$\begin{aligned}
\sum_{i=1}^k |\mathcal{A}_i| &= \sum_{i=1}^k |{\mathcal{A}_i}^{t,-}| + \sum_{i=1}^k |{\mathcal{A}_i}^{t,+}| \\
&\leq |\mathcal{A}^{t,-}| + k|\mathcal{A}^{t,+}| \qquad \text{(by Lemma 2.1)} \\
&= k\left(|\mathcal{A}^{t,+}| + \frac{1}{k}|\mathcal{A}^{t,-}|\right) \\
&\leq k\left(|\mathcal{A}^{t,+}| + \beta(\mathcal{F}, t)|\mathcal{A}^{t,-}|\right) \\
&\leq k(l(\mathcal{F}, t)) \qquad \text{(by (1))}
\end{aligned} \tag{4}$$

and, by Lemma 1.3 and (4),

$$\prod_{i=1}^k |\mathcal{A}_i| \leq \left(\frac{1}{k}\sum_{i=1}^k |\mathcal{A}_i|\right)^k \leq (l(\mathcal{F}, t))^k. \tag{5}$$

If $\mathcal{A}_1 = ... = \mathcal{A}_k = \mathcal{L}$ for some largest t-intersecting subfamily $\mathcal{L}$ of $\mathcal{F}$, then obviously $\mathcal{A}_1, ..., \mathcal{A}_k$ are cross-t-intersecting, $\sum_{i=1}^{k} |\mathcal{A}_i| = k(l(\mathcal{F},t))$ and $\prod_{i=1}^{k} |\mathcal{A}_i| = (l(\mathcal{F},t))^k$. Now suppose $k > \kappa(\mathcal{F},t)$ and either $\sum_{i=1}^{k} |\mathcal{A}_i| = k(l(\mathcal{F},t))$ or $\prod_{i=1}^{k} |\mathcal{A}_i| = (l(\mathcal{F},t))^k$. If $\prod_{i=1}^{k} |\mathcal{A}_i| = (l(\mathcal{F},t))^k$, then $\sum_{i=1}^{k} |\mathcal{A}_i| = k(l(\mathcal{F},t))$ by (5). So $\sum_{i=1}^{k} |\mathcal{A}_i| = k(l(\mathcal{F},t))$. Thus in (4) we have equality throughout. It follows that $|\mathcal{A}^{t,-}| = 0$ (since $k > \kappa(\mathcal{F},t)$ implies that $\frac{1}{k} < \beta(\mathcal{F},t)$), and hence $\mathcal{A} = \mathcal{A}^{t,+}$. So $\mathcal{A}$ is a t-intersecting subfamily of $\mathcal{F}$. Since $\sum_{i=1}^{k} |\mathcal{A}_i| = k(l(\mathcal{F},t))$ and $\mathcal{A}_i \subseteq \mathcal{A}$ for each $i \in [k]$, it follows that $\mathcal{A}_1 = ... = \mathcal{A}_k = \mathcal{A}$ and $\mathcal{A}$ is a largest t-intersecting subfamily of $\mathcal{F}$. $\Box$

3 The parameter $\beta(\mathcal{F},t)$

Theorems 1.1 and 1.2 tell us that $\lceil \kappa(\mathcal{F},t) \rceil$ is the smallest integer k_0 such that for any $k \geq k_0$, the configuration $\mathcal{A}_1 = ... = \mathcal{A}_k = \mathcal{L}$ (as in the theorems) is optimal for the maximisation of both the sum and the product of sizes. So $\kappa(\mathcal{F},t)$ is an important parameter and hence worth investigating. But $\kappa(\mathcal{F},t)$ is simply defined to be the reciprocal of $\beta(\mathcal{F},t)$, and hence we may instead focus on $\beta(\mathcal{F},t)$.

In this section we first establish some basic facts on $\beta(\mathcal{F},t)$ and then we provide the value of $\beta(\mathcal{F},t)$ for various important families $\mathcal{F}$.

3.1 General facts

We start by proving the lower bound in (3) and characterising the families for which it is attained.

Proposition 3.1 *For any family $\mathcal{F} \neq \emptyset$,*

$$\beta(\mathcal{F},t) \geq \frac{1}{|\mathcal{F}|},$$

and equality holds if and only if $|A \cap B| < t$ for any distinct A and B in $\mathcal{F}$.

Proof. Since $\mathcal{F}$ is non-empty, $l(\mathcal{F},t) \geq 1$ because any subfamily of $\mathcal{F}$ consisting of only one set is t-intersecting (by definition). Let $\mathcal{A} \subseteq \mathcal{F}$. If $\mathcal{A}^{t,-} = \emptyset$, then $\beta(\mathcal{F},t,\mathcal{A}) = \frac{l(\mathcal{F},t)}{|\mathcal{F}|} \geq \frac{1}{|\mathcal{F}|}$, and equality holds only if $l(\mathcal{F},t) = 1$. Now suppose $\mathcal{A}^{t,-} \neq \emptyset$. Let $A \in \mathcal{A}^{t,-}$. Then $\mathcal{A}^{t,+} \cup \{A\}$ is a t-intersecting subfamily of $\mathcal{F}$, and hence $|\mathcal{A}^{t,+} \cup \{A\}| \leq l(\mathcal{F},t)$. So $l(\mathcal{F},t) \geq |\mathcal{A}^{t,+}| + 1$. We therefore have $\beta(\mathcal{F},t,\mathcal{A}) = \frac{l(\mathcal{F},t) - |\mathcal{A}^{t,+}|}{|\mathcal{A}^{t,-}|} \geq \frac{1}{|\mathcal{F}|}$, and equality holds only if $l(\mathcal{F},t) - |\mathcal{A}^{t,+}| = 1$ and $\mathcal{A}^{t,-} = \mathcal{F}$, in which case $\mathcal{A}^{t,+} = \emptyset$ and hence $l(\mathcal{F},t) = 1$.

Therefore, $\beta(\mathcal{F},t) \geq \frac{1}{|\mathcal{F}|}$, and equality holds only if $l(\mathcal{F},t) = 1$. Now clearly $l(\mathcal{F},t) = 1$ if and only if $|A \cap B| < t$ for any distinct A and B in $\mathcal{F}$, in which case either $|\mathcal{F}| = 1$ or for all $\mathcal{A} \subseteq \mathcal{F}$ with $|\mathcal{A}| \geq 2$, $\mathcal{A}^{t,-} = \mathcal{A}$ and $\mathcal{A}^{t,+} = \emptyset$. So $l(\mathcal{F},t) = 1$ implies that $\beta(\mathcal{F},t) = \beta(\mathcal{F},t,\mathcal{F}) = \frac{1}{|\mathcal{F}|}$. $\Box$

We next prove (2), which gives a clear description of $\beta(\mathcal{F},t)$, and hence establish the upper bound in (3).

Proposition 3.2 *For any family $\mathcal{F} \neq \emptyset$, $\beta(\mathcal{F},t)$ is the largest real number $c \leq \frac{l(\mathcal{F},t)}{|\mathcal{F}|}$ such that*

$$|\mathcal{A}^{t,+}| + c|\mathcal{A}^{t,-}| \leq l(\mathcal{F},t) \quad \textit{for any } \mathcal{A} \subseteq \mathcal{F}.$$

Proof. For any $\mathcal{A} \subseteq \mathcal{F}$, $|\mathcal{A}^{t,+}| + \beta(\mathcal{F},t)|\mathcal{A}^{t,-}| \leq |\mathcal{A}^{t,+}| + \beta(\mathcal{F},t,\mathcal{A})|\mathcal{A}^{t,-}| \leq l(\mathcal{F},t)$. Since $\beta(\mathcal{F},t,\emptyset) = \frac{l(\mathcal{F},t)}{|\mathcal{F}|}$,

$$\beta(\mathcal{F},t) \leq \frac{l(\mathcal{F},t)}{|\mathcal{F}|}.$$

Suppose $\beta(\mathcal{F},t) < \frac{l(\mathcal{F},t)}{|\mathcal{F}|}$. Let d be a real number greater than $\beta(\mathcal{F},t)$. Let $\mathcal{A}_0 \subseteq \mathcal{F}$ such that $\beta(\mathcal{F},t,\mathcal{A}_0) = \beta(\mathcal{F},t)$. Since $\beta(\mathcal{F},t,\mathcal{A}_0) \neq \frac{l(\mathcal{F},t)}{|\mathcal{F}|}$, we have ${\mathcal{A}_0}^{t,-} \neq \emptyset$ and hence $|{\mathcal{A}_0}^{t,+}| + \beta(\mathcal{F},t,\mathcal{A}_0)|{\mathcal{A}_0}^{t,-}| = l(\mathcal{F},t)$. So $|{\mathcal{A}_0}^{t,+}| + d|{\mathcal{A}_0}^{t,-}| > l(\mathcal{F},t)$. Hence the result. $\Box$

Remark 3.3 When $\mathcal{A}^{t,-} = \emptyset$ it does not matter what $\beta(\mathcal{F},t,\mathcal{A})$ is, because $|\mathcal{A}^{t,+}| + \beta(\mathcal{F},t,\mathcal{A})|\mathcal{A}^{t,-}| = |\mathcal{A}^{t,+}| \leq l(\mathcal{F},t)$. Thus we could define $\beta(\mathcal{F},t)$ to be the minimum value of $\beta(\mathcal{F},t,\mathcal{A})$ such that $\mathcal{A}^{t,-} \neq \emptyset$ when such a subfamily $\mathcal{A}$ exists, i.e. when $\mathcal{F}^{t,-} \neq \emptyset$ (i.e. when $\mathcal{F}$ is not t-intersecting). However, this would still give us $\beta(\mathcal{F},t) \leq \frac{l(\mathcal{F},t)}{|\mathcal{F}|}$; indeed,

$$\begin{aligned} \mathcal{F}^{t,-} \neq \emptyset \;\Rightarrow\; \beta(\mathcal{F},t,\mathcal{F}) &= \frac{l(\mathcal{F},t) - |\mathcal{F}^{t,+}|}{|\mathcal{F}^{t,-}|} = \frac{l(\mathcal{F},t) - |\mathcal{F}^{t,+}|}{|\mathcal{F}| - |\mathcal{F}^{t,+}|} \leq \frac{l(\mathcal{F},t)}{|\mathcal{F}|} \qquad (6) \\ \Rightarrow\; \beta(\mathcal{F},t) &\leq \frac{l(\mathcal{F},t)}{|\mathcal{F}|}. \end{aligned}$$

If $\mathcal{F}^{t,-} \neq \emptyset$ and $\mathcal{F}^{t,+} \neq \emptyset$, then the inequality in (6) is strict. Thus, if $\beta(\mathcal{F},t) = \frac{l(\mathcal{F},t)}{|\mathcal{F}|}$, then either $\mathcal{F}^{t,-} = \emptyset$ or $\mathcal{F}^{t,+} = \emptyset$. Therefore,

$$\beta(\mathcal{F},t) = \frac{l(\mathcal{F},t)}{|\mathcal{F}|} \;\Rightarrow\; \mathcal{F} = \mathcal{F}^{t,+} \text{ or } \mathcal{F} = \mathcal{F}^{t,-}. \qquad (7)$$

Example 3.4 Let $F_1, ..., F_n$ be $n \geq 2$ disjoint sets, each of size at least t, and let $F_{n+1} = \bigcup_{i=1}^{n} F_i$. Let $\mathcal{F} = \{F_1, ..., F_n, F_{n+1}\}$. Then $\mathcal{F}^{t,+} = \{F_{n+1}\}$, $\mathcal{F}^{t,-} = \{F_1, ..., F_n\}$ and $l(\mathcal{F},t) = 2$. By (7), $\beta(\mathcal{F},t) \neq \frac{l(\mathcal{F},t)}{|\mathcal{F}|}$ (so $\beta(\mathcal{F},t) < \frac{l(\mathcal{F},t)}{|\mathcal{F}|}$ by (3)); one can easily check that in fact $\beta(\mathcal{F},t) = \beta(\mathcal{F},t,\mathcal{F}) = \frac{1}{n}$, and hence $\frac{1}{|\mathcal{F}|} < \beta(\mathcal{F},t) < \frac{l(\mathcal{F},t)}{|\mathcal{F}|}$.

Clearly, if $\mathcal{F} = \mathcal{F}^{t,+}$, then $\mathcal{F}$ is t-intersecting and hence $\beta(\mathcal{F},t) = \frac{l(\mathcal{F},t)}{|\mathcal{F}|} = 1$. However, if $\mathcal{F} = \mathcal{F}^{t,-}$, then we do not necessarily have $\beta(\mathcal{F},t) = \frac{l(\mathcal{F},t)}{|\mathcal{F}|}$; so the converse of (7) is not true.

Example 3.5 Let $2 \leq m < n$, and let $F_1, ..., F_n, F_{n+1}$ be as in Example 3.4. Let $F_{n+2}, ..., F_{n+m}$ be sets that are disjoint from each other and from $F_{n+1} = \bigcup_{i=1}^{n+1} F_i$. Let $\mathcal{F} = \{F_1, ..., F_{n+m}\}$. Then $\mathcal{F} = \mathcal{F}^{t,-}$ and $l(\mathcal{F},t) = 2$. Let $\mathcal{A} = \{F_1, ..., F_{n+1}\}$. Then $\mathcal{A}^{t,+} = \{F_{n+1}\}$, $\mathcal{A}^{t,-} = \{F_1, ..., F_n\}$, and $\beta(\mathcal{F},t) \leq \beta(\mathcal{F},t,\mathcal{A}) = \frac{1}{n} < \frac{2}{n+m} = \frac{l(\mathcal{F},t)}{|\mathcal{F}|}$.

3.2 The value of $\beta(\mathcal{F},t)$ for various important families $\mathcal{F}$

There are many important families $\mathcal{F}$ which attain the upper bound in (3).

In each of the papers [5, 6, 7, 12], a particular important family $\mathcal{F}$ is considered, and it is proved that $|\mathcal{A}^{1,+}| + \frac{l(\mathcal{F},1)}{|\mathcal{F}|}|\mathcal{A}^{1,-}| \leq l(\mathcal{F},1)$ for any $\mathcal{A} \subseteq \mathcal{F}$, meaning that $\beta(\mathcal{F},1) = \frac{l(\mathcal{F},1)}{|\mathcal{F}|}$ by Proposition 3.2. In [5] (a paper inspired by [20]), $\mathcal{F}$ is $\binom{[n]}{r}$, $r \leq n/2$ (if $n/2 < r \leq n$, then $\mathcal{F}$ is 1-intersecting, and hence $\beta(\mathcal{F},1) = \frac{l(\mathcal{F},1)}{|\mathcal{F}|}$ still holds). In [6], $\mathcal{F}$ is

$$\mathcal{P}_{r,n} = \{\{(1,y_1),(2,y_2),...,(r,y_r)\} \colon y_1,y_2,...,y_r \text{ are distinct elements of } [n]\} \quad (r \in [n]),$$

which describes permutations of r-subsets of $[n]$ (see [6]). In [7], $\mathcal{F}$ is

$$\mathcal{P}_n^{(r)} = \{\{(x_1, y_1), ..., (x_r, y_r)\} \colon x_1, ..., x_r \text{ are distinct elements of } [n], \\ y_1, ..., y_r \text{ are distinct elements of } [n]\} \quad (r \in [n]),$$

which describes r-partial permutations of $[n]$ (see [7]). In [12], $\mathcal{F}$ is the family

$$\mathcal{S}_{n,r,m} = \{\{(x_1, y_1), ..., (x_r, y_r)\} \colon x_1, ..., x_r \text{ are distinct elements of } [n],\ y_1, ..., y_r \in [m]\}$$

of m-signed r-subsets of $[n]$, where $r \in [n]$ and $m \geq 2$. For each of these families, the value of $l(\mathcal{F}, 1)$ is known and is attained by the intersecting subfamily $\{F \in \mathcal{F} \colon x \in F\}$ for any $x \in \bigcup_{F \in \mathcal{F}} F$ (for example, the subfamily $\left\{A \in \binom{[n]}{r} \colon 1 \in A\right\}$ of $\binom{[n]}{r}$, the subfamily $\{A \in \mathcal{P}_{r,n} \colon (1, 1) \in A\}$ of $\mathcal{P}_{r,n}$, etc.); see [10].

We now prove that the same holds for the *power set* of a set X, i.e. the family of all subsets of X, which is perhaps the most natural family one can think of. Let 2^X denote the power set of X. One of the basic results in extremal set theory is that $l(2^{[n]}, 1) = 2^{n-1}$ (see [17]), and this is generalised by our next result.

Theorem 3.6 *If $\mathcal{F} = 2^{[n]}$, then*

$$\beta(\mathcal{F}, 1) = \frac{l(\mathcal{F}, 1)}{|\mathcal{F}|} = \frac{1}{2}.$$

Proof. Let $\mathcal{A} \subseteq \mathcal{F} = 2^{[n]}$. Let $\mathcal{B} = \{[n] \backslash A \colon A \in \mathcal{A}^{1,+}\}$. So $|\mathcal{B}| = |\mathcal{A}^{1,+}|$. For any $B \in \mathcal{B}$, we have $B = [n] \backslash A$ for some $A \in \mathcal{A}^{1,+}$, and hence, by definition of $\mathcal{A}^{1,+}$, $B \notin \mathcal{A}$ since $|A \cap B| = 0$. So $\mathcal{A}$ and $\mathcal{B}$ are disjoint subfamilies of $\mathcal{F}$. Therefore,

$$2|\mathcal{A}^{1,+}| + |\mathcal{A}^{1,-}| = |\mathcal{A}^{1,+}| + |\mathcal{B}| + |\mathcal{A}^{1,-}| = |\mathcal{A}| + |\mathcal{B}| = |\mathcal{A} \cup \mathcal{B}| \leq |\mathcal{F}| = 2^n$$

and hence, dividing throughout by 2, we get $|\mathcal{A}^{1,+}| + \frac{1}{2}|\mathcal{A}^{1,-}| \leq 2^{n-1}$. It follows that the size of any 1-intersecting subfamily of $\mathcal{F}$ is at most 2^{n-1} (as $\mathcal{A} = \mathcal{A}^{1,+}$ if $\mathcal{A}$ is 1-intersecting), and this bound is attained by the trivial 1-intersecting subfamily $\{A \in \mathcal{F} \colon 1 \in A\}$; so $l(\mathcal{F}, 1) = 2^{n-1}$ and $\frac{l(\mathcal{F},1)}{|\mathcal{F}|} = \frac{1}{2}$. So we have $|\mathcal{A}^{1,+}| + \frac{l(\mathcal{F},1)}{|\mathcal{F}|}|\mathcal{A}^{1,-}| \leq l(\mathcal{F}, 1)$. By Proposition 3.2, $\beta(\mathcal{F}, 1) = \frac{l(\mathcal{F},1)}{|\mathcal{F}|} = \frac{1}{2}$. □

Note that by Theorems 3.6 and 1.1, for $\mathcal{F} = 2^{[n]}$, the configuration $\mathcal{A}_1 = ... = \mathcal{A}_k = \mathcal{L}$ is optimal for both the sum and the product for any $k \geq 2$. More precisely, we have the following.

Theorem 3.7 *Let $k \geq 2$, and let $\mathcal{A}_1, ..., \mathcal{A}_k$ be cross-1-intersecting subfamilies of $2^{[n]}$. Then*

$$\sum_{i=1}^{k} |\mathcal{A}_i| \leq k2^{n-1} \quad \text{and} \quad \prod_{i=1}^{k} |\mathcal{A}_i| \leq 2^{k(n-1)},$$

and both bounds are attained if $\mathcal{A}_1 = ... = \mathcal{A}_k = \{A \in 2^{[n]} \colon 1 \in A\}$. Moreover, if $k > 2$, then in both inequalities, equality holds if and only if $\mathcal{A}_1 = ... = \mathcal{A}_k = \mathcal{L}$ for some largest 1-intersecting subfamily $\mathcal{L}$ of $\mathcal{F}$.[1]

[1] At the time of writing this paper, this result was generalised in [8] for any union of power sets of sets which have a common element.

The above results for $\beta(\mathcal{F},1)$ generalise for $\beta(\mathcal{F},t)$ as follows. Recently, Wang and Zhang [29] observed that the method employed in [5, 6, 7, 12] together with a result for vertex-transitive graphs found in [3] and also in [13] (see [29]) immediately give us $\beta(\mathcal{F},t) = \frac{l(\mathcal{F},t)}{|\mathcal{F}|}$ for the following very important class of families.

We shall call a family $\mathcal{F}$ *t-symmetric* if there exists a group Γ of bijections with domain $\mathcal{F}$ and co-domain $\mathcal{F}$ such that Γ acts transitively on $\mathcal{F}$ and preserves the t-intersection property, i.e. for any $A, B \in \mathcal{F}$, the following hold:
(a) there exists $\delta \in \Gamma$ such that $B = \delta(A)$;
(b) if A t-intersects B, then for all $\gamma \in \Gamma$, $\gamma(A)$ t-intersects $\gamma(B)$.

The result proved by Wang and Zhang [29, Corollary 2.4] gives us the following.

Theorem 3.8 ([29]) *If $\mathcal{A}$ is a subfamily of a t-symmetric family $\mathcal{F}$, then*

$$|\mathcal{A}^{t,+}| + \frac{l(\mathcal{F},t)}{|\mathcal{F}|}|\mathcal{A}^{t,-}| \leq l(\mathcal{F},t).$$

Together with Proposition 3.2, this immediately gives us the next result.

Corollary 3.9 *For any t-symmetric family $\mathcal{F}$,*

$$\beta(\mathcal{F},t) = \frac{l(\mathcal{F},t)}{|\mathcal{F}|}.$$

It turns out that the families $\binom{[n]}{r}$, $\mathcal{P}_{r,n}$, $\mathcal{P}_n^{(r)}$ and $\mathcal{S}_{n,r,m}$ are t-symmetric for any $t \geq 1$. Also, the value of $l(\mathcal{F},t)$ is known precisely for the following cases: $\mathcal{F} = \binom{[n]}{r}$ (see [1]), $\mathcal{F} = \mathcal{S}_{n,n,m}$ (see [2, 19]), $\mathcal{F} = \mathcal{S}_{n,r,m}$ with $n \geq (r-t+m)(t+1)/m$ (see [4]), $\mathcal{F} = \mathcal{P}_{n,n}$ with n sufficiently larger than t (see [16]), $\mathcal{F} = \mathcal{P}_{r,n}$ with n sufficiently larger than r (see [11]), and $\mathcal{P}_n^{(r)}$ with n sufficiently larger than r (see [24, 11, 9]). Thus, by Corollary 3.9, we know $\beta(\mathcal{F},t)$ for each of these cases.

Another important family for which we have similar results is the family $\mathcal{V}_{n,r}(q)$ of all r-dimensional subspaces of an n-dimensional vector space over a q-element field; however, for this family, t-intersection is defined slightly differently, and we will discuss this separately in Section 4.2.

Now $l(2^{[n]},t)$ was determined in [22], and although $2^{[n]}$ is not t-symmetric, we will now determine $\beta(2^{[n]},t)$ using the fact that $\mathcal{S}_{n,n,2}$ is t-symmetric and that we also know $l(\mathcal{S}_{n,n,2},t)$ (see [23], and see [2, 19] for $\mathcal{S}_{n,n,m}$), which is in fact equal to $l(2^{[n]},t)$. Define

$$\mathcal{K}_{n,t} = \begin{cases} \{A \subseteq [n] \colon |A| \geq (n+t)/2\} & \text{if } n+t \text{ is even;} \\ \{A \subseteq [n] \colon |A \cap [n-1]| \geq (n+t-1)/2\} & \text{if } n+t \text{ is odd.} \end{cases}$$

Katona [22] proved that $\mathcal{K}_{n,t}$ is a largest t-intersecting subfamily of $2^{[n]}$ (and uniquely so up to isomorphism if $t \geq 2$); so $l(2^{[n]},t) = |\mathcal{K}_{n,t}|$. Kleitman [23] showed that we also have $l(\mathcal{S}_{n,n,2},t) = |\mathcal{K}_{n,t}|$.

Theorem 3.10 *If $\mathcal{F} = 2^{[n]}$ and $n \geq t$, then*

$$\beta(\mathcal{F},t) = \frac{l(\mathcal{F},t)}{|\mathcal{F}|} = \frac{|\mathcal{K}_{n,t}|}{2^n}.$$

Proof. Let $\mathcal{A} \subseteq 2^{[n]}$. For each $A \in \mathcal{A}$, let B_A be the set $\{(a,1)\colon a \in A\} \cup \{(b,2)\colon b \in [n]\backslash A\}$ in $\mathcal{S}_{n,n,2}$. Let $\mathcal{B}$ be the subfamily $\{B_A\colon A \in \mathcal{A}\}$ of $\mathcal{S}_{n,n,2}$. By Theorem 3.8, $|\mathcal{B}^{t,+}| + \frac{l(\mathcal{S}_{n,n,2},t)}{|\mathcal{S}_{n,n,2}|}|\mathcal{B}^{t,-}| \leq l(\mathcal{S}_{n,n,2},t)$. Since $|\mathcal{S}_{n,n,2}| = 2^n$ and $l(\mathcal{S}_{n,n,2},t) = |\mathcal{K}_{n,t}|$, we get $|\mathcal{B}^{t,+}| + \frac{|\mathcal{K}_{n,t}|}{2^n}|\mathcal{B}^{t,-}| \leq |\mathcal{K}_{n,t}|$. Now we clearly have that if $A \in \mathcal{A}^{t,+}$, then $B_A \in \mathcal{B}^{t,+}$. So $|\mathcal{A}^{t,+}| = |\mathcal{B}^{t,+}| - p$ for some non-negative integer p, and hence, since $|\mathcal{A}^{t,+}| + |\mathcal{A}^{t,-}| = |\mathcal{A}| = |\mathcal{B}| = |\mathcal{B}^{t,+}| + |\mathcal{B}^{t,-}|$, we have $|\mathcal{A}^{t,-}| = |\mathcal{B}^{t,-}| + p$. So we have

$$|\mathcal{A}^{t,+}| + \frac{l(2^{[n]},t)}{|2^{[n]}|}|\mathcal{A}^{t,-}| = (|\mathcal{B}^{t,+}| - p) + \frac{|\mathcal{K}_{n,t}|}{2^n}(|\mathcal{B}^{t,-}| + p) \leq |\mathcal{B}^{t,+}| + \frac{|\mathcal{K}_{n,t}|}{2^n}|\mathcal{B}^{t,-}| \leq |\mathcal{K}_{n,t}|.$$

Therefore, by Proposition 3.2, $\beta(2^{[n]},t) = \frac{l(2^{[n]},t)}{|2^{[n]}|}$ and hence the result. □

4 The maximum sum

In this section we restrict our attention to the the problem of maximising the sum of sizes of *any* number of cross-t-intersecting subfamilies of a given family $\mathcal{F}$. Similarly to Section 3, we first prove general results and then we provide complete solutions for various important families.

4.1 General results and observations

We start by proving Theorem 1.2.

Proof of Theorem 1.2. (i) is given by Theorem 1.1. Suppose $k < \kappa(\mathcal{F},t)$. So $\frac{1}{k} > \beta(\mathcal{F},t)$.

Case 1: $\beta(\mathcal{F},t) = \frac{l(\mathcal{F},t)}{|\mathcal{F}|}$. So $k < \frac{|\mathcal{F}|}{l(\mathcal{F},t)}$ and hence $k(l(\mathcal{F},t)) < |\mathcal{F}|$. Let $\mathcal{B}_1 = \mathcal{F}$ and $\mathcal{B}_2 = ... = \mathcal{B}_k = \emptyset$. Since $\mathcal{B}_1, ..., \mathcal{B}_k$ are cross-t-intersecting, $\sum_{i=1}^k |\mathcal{A}_i| \geq \sum_{i=1}^k |\mathcal{B}_i|$. So we have $\sum_{i=1}^k |\mathcal{A}_i| \geq |\mathcal{F}| > k(l(\mathcal{F},t))$.

Case 2: $\beta(\mathcal{F},t) \neq \frac{l(\mathcal{F},t)}{|\mathcal{F}|}$. By (3), $\beta(\mathcal{F},t) < \frac{l(\mathcal{F},t)}{|\mathcal{F}|}$. Thus, taking $\mathcal{A}_0 \subseteq \mathcal{F}$ such that $\beta(\mathcal{F},t,\mathcal{A}_0) = \beta(\mathcal{F},t)$, we have ${\mathcal{A}_0}^{t,-} \neq \emptyset$ and $|{\mathcal{A}_0}^{t,+}| + \beta(\mathcal{F},t)|{\mathcal{A}_0}^{t,-}| = l(\mathcal{F},t)$. Since $\frac{1}{k} > \beta(\mathcal{F},t)$, $|{\mathcal{A}_0}^{t,+}| + \frac{1}{k}|{\mathcal{A}_0}^{t,-}| > l(\mathcal{F},t)$. Let $\mathcal{B}_1 = \mathcal{A}_0$ and $\mathcal{B}_2 = ... = \mathcal{B}_k = {\mathcal{A}_0}^{t,+}$. Then

$$\sum_{i=1}^k |\mathcal{B}_i| = (|{\mathcal{A}_0}^{t,-}| + |{\mathcal{A}_0}^{t,+}|) + (k-1)|{\mathcal{A}_0}^{t,+}| = k\left(|{\mathcal{A}_0}^{t,+}| + \frac{1}{k}|{\mathcal{A}_0}^{t,-}|\right)$$

and hence $\sum_{i=1}^k |\mathcal{B}_i| > k(l(\mathcal{F},t))$. Since $\mathcal{B}_1, ..., \mathcal{B}_k$ are cross-t-intersecting, we have $\sum_{i=1}^k |\mathcal{A}_i| \geq \sum_{i=1}^k |\mathcal{B}_i| > k(l(\mathcal{F},t))$. □

As we have seen in Section 3, the case $\beta(\mathcal{F},t) = \frac{l(\mathcal{F},t)}{|\mathcal{F}|}$ deserves very special attention. For this particularly interesting case, we have the following precise result, which gives us the maximum sum of sizes for any $k \geq 2$, and characterises optimal configurations. Recall that $\max\{|F|\colon F \in \mathcal{F}\}$ is denoted by $\alpha(\mathcal{F})$.

Theorem 4.1 *Let $\mathcal{F}$ be a family with $\alpha(\mathcal{F}) \geq t$ and $\beta(\mathcal{F},t) = \frac{l(\mathcal{F},t)}{|\mathcal{F}|}$. Let $\mathcal{A}_1, ..., \mathcal{A}_k$ be cross-t-intersecting subfamilies of $\mathcal{F}$ such that $\sum_{i=1}^k |\mathcal{A}_i|$ is maximum. Then*

$$\sum_{i=1}^k |\mathcal{A}_i| = \begin{cases} |\mathcal{F}| & \text{if } k \leq \frac{|\mathcal{F}|}{l(\mathcal{F},t)}; \\ k(l(\mathcal{F},t)) & \text{if } k \geq \frac{|\mathcal{F}|}{l(\mathcal{F},t)}. \end{cases}$$

Moreover,
(i) if $k < \frac{|\mathcal{F}|}{l(\mathcal{F},t)}$, *then* $\mathcal{A}_i = {\mathcal{A}_i}^{t,-}$ *for all* $i \in [k]$, *and* $\mathcal{A}_1, ..., \mathcal{A}_k$ *partition* $\mathcal{F}$*;*
(ii) if $k > \frac{|\mathcal{F}|}{l(\mathcal{F},t)}$, *then* $\mathcal{A}_1 = ... = \mathcal{A}_k = \mathcal{L}$ *for some largest* t*-intersecting subfamily* $\mathcal{L}$ *of* $\mathcal{F}$*.*

Remark 4.2 An optimal configuration for the case $k \leq \frac{|\mathcal{F}|}{l(\mathcal{F},t)}$ is the one with $\mathcal{A}_1 = \mathcal{F}$ and $\mathcal{A}_2 = ... = \mathcal{A}_k = \emptyset$; we will call this the *trivial configuration*. If $k = \frac{|\mathcal{F}|}{l(\mathcal{F},t)}$, then the configuration $\mathcal{A}_1 = ... = \mathcal{A}_k = \mathcal{L}$ is also optimal. For each of the cases $k < \frac{|\mathcal{F}|}{l(\mathcal{F},t)}$ and $k = \frac{|\mathcal{F}|}{l(\mathcal{F},t)}$, it is possible to have other optimal configurations but it is also possible to not have any others; [6, Theorem 1.4] gives an example of each of these possibilities for $t = 1$.

Proof of Theorem 4.1. Since $\beta(\mathcal{F},t) = \frac{l(\mathcal{F},t)}{|\mathcal{F}|}$, $\kappa(\mathcal{F},t) = \frac{|\mathcal{F}|}{l(\mathcal{F},t)}$. So the case $k \geq \frac{|\mathcal{F}|}{l(\mathcal{F},t)}$ is given by Theorem 1.1.

Let $\mathcal{A} = \bigcup_{i=1}^k \mathcal{A}_i$. Lemma 2.1 tells us that $\mathcal{A}^{t,+} = \bigcup_{i=1}^k {\mathcal{A}_i}^{t,+}$, $\mathcal{A}^{t,-} = \bigcup_{i=1}^k {\mathcal{A}_i}^{t,-}$, and ${\mathcal{A}_1}^{t,-}, ..., {\mathcal{A}_k}^{t,-}$ partition $\mathcal{A}^{t,-}$. So we have

$$\begin{aligned}
\sum_{i=1}^k |\mathcal{A}_i| &= \sum_{i=1}^k |{\mathcal{A}_i}^{t,-}| + \sum_{i=1}^k |{\mathcal{A}_i}^{t,+}| \leq |\mathcal{A}^{t,-}| + k|\mathcal{A}^{t,+}| \\
&\leq \frac{1}{\beta(\mathcal{F},t)}(l(\mathcal{F},t) - |\mathcal{A}^{t,+}|) + k|\mathcal{A}^{t,+}| \qquad \text{(by (1))} \\
&= \frac{|\mathcal{F}|}{l(\mathcal{F},t)}(l(\mathcal{F},t) - |\mathcal{A}^{t,+}|) + k|\mathcal{A}^{t,+}| = |\mathcal{F}| + \left(k - \frac{|\mathcal{F}|}{l(\mathcal{F},t)}\right)|\mathcal{A}^{t,+}| \qquad (8)
\end{aligned}$$

(note that the case $k \geq \frac{|\mathcal{F}|}{l(\mathcal{F},t)}$ can be deduced from (8) since $|\mathcal{A}^{t,+}| \leq l(\mathcal{F},t)$). Suppose $k \leq \frac{|\mathcal{F}|}{l(\mathcal{F},t)}$. Then $\sum_{i=1}^k |\mathcal{A}_i| \leq |\mathcal{F}|$, and if $k < \frac{|\mathcal{F}|}{l(\mathcal{F},t)}$, then, by (8), the bound is attained only if $\mathcal{A}^{t,+} = \emptyset$ and $\mathcal{A}^{t,-} = \mathcal{F}$, which implies that $\mathcal{A}_i = {\mathcal{A}_i}^{t,-}$ for all $i \in [k]$, and that $\mathcal{A}_1, ..., \mathcal{A}_k$ partition $\mathcal{F}$. Now let $\mathcal{B}_1 = \mathcal{F}$ and $\mathcal{B}_2 = ... = \mathcal{B}_k = \emptyset$. Since $\mathcal{B}_1, ..., \mathcal{B}_k$ are cross-t-intersecting and $\sum_{i=1}^k |\mathcal{B}_i| = |\mathcal{F}|$, we have $|\mathcal{F}| \leq \sum_{i=1}^k |\mathcal{A}_i|$ (as $\sum_{i=1}^k |\mathcal{A}_i|$ is maximum). Together with $\sum_{i=1}^k |\mathcal{A}_i| \leq |\mathcal{F}|$, this gives us $\sum_{i=1}^k |\mathcal{A}_i| = |\mathcal{F}|$. Hence the result. □

The results above raise the following question: can we say something in general about the structure of an optimal configuration for $k < \kappa(\mathcal{F},t)$? An answer is given by the next result, which in particular describes an optimal configuration.

Proposition 4.3 *Let* $\mathcal{F}$ *and* $\mathcal{A}_1, ..., \mathcal{A}_k$ *be as in Theorem 1.2. Let* $\mathcal{A} = \bigcup_{i=1}^k \mathcal{A}_i$*. Let* $\mathcal{A}_0$ *be a subfamily of* $\mathcal{F}$ *such that* $|{\mathcal{A}_0}^{t,+}| + \frac{1}{k}|{\mathcal{A}_0}^{t,-}|$ *is maximum, and let* $\mathcal{B}_1 = \mathcal{A}_0$ *and* $\mathcal{B}_2 = ... = \mathcal{B}_k = {\mathcal{A}_0}^{t,+}$*. Then*
(i) $\mathcal{B}_1, ..., \mathcal{B}_k$ *are cross-*t*-intersecting subfamilies of* $\mathcal{F}$*,*
(ii) $\sum_{i=1}^k |\mathcal{A}_i| = \sum_{i=1}^k |\mathcal{B}_i|$ *and* $|\mathcal{A}^{t,+}| + \frac{1}{k}|\mathcal{A}^{t,-}| = |{\mathcal{A}_0}^{t,+}| + \frac{1}{k}|{\mathcal{A}_0}^{t,-}|$*.*

Proof. (i) is trivial. As in the proof of Theorem 1.1, $\sum_{i=1}^k |\mathcal{A}_i| \leq k\left(|\mathcal{A}^{t,+}| + \frac{1}{k}|\mathcal{A}^{t,-}|\right)$. Thus, by the choice of $\mathcal{A}_0$, $\sum_{i=1}^k |\mathcal{A}_i| \leq k\left(|{\mathcal{A}_0}^{t,+}| + \frac{1}{k}|{\mathcal{A}_0}^{t,-}|\right) = \sum_{i=1}^k |\mathcal{B}_i|$, where the equality follows as in the proof of Theorem 1.2. Now by (i) and the choice of $\mathcal{A}_1, ..., \mathcal{A}_k$, $\sum_{i=1}^k |\mathcal{A}_i| \geq \sum_{i=1}^k |\mathcal{B}_i|$. So we actually have $\sum_{i=1}^k |\mathcal{A}_i| = \sum_{i=1}^k |\mathcal{B}_i|$ and hence $|\mathcal{A}^{t,+}| + \frac{1}{k}|\mathcal{A}^{t,-}| = |{\mathcal{A}_0}^{t,+}| + \frac{1}{k}|{\mathcal{A}_0}^{t,-}|$. □

Remark 4.4 We know from Theorems 1.1 and 1.2 that the configuration $\mathcal{A}_1, ..., \mathcal{A}_k = \mathcal{L}$ is always optimal for $k \geq \kappa(\mathcal{F},t)$ (and uniquely so if $k > \kappa(\mathcal{F},t)$) and never optimal for $k < \kappa(\mathcal{F}, t)$. Theorem 4.1 tells us that the trivial configuration (see Remark 4.2) is always optimal for $k < \kappa(\mathcal{F}, t)$ if $\beta(\mathcal{F}, t) = \frac{l(\mathcal{F},t)}{|\mathcal{F}|}$. However, as Proposition 4.3 suggests, the trivial configuration may not be optimal for $k < \kappa(\mathcal{F}, t)$ if $\beta(\mathcal{F}, t) \neq \frac{l(\mathcal{F},t)}{|\mathcal{F}|}$, meaning that it is possible to have $\mathcal{F}$ and k for which neither of the two simple configurations mentioned in Remark 4.2 give a maximum sum.

Proposition 4.5 *Let $\mathcal{F}$ be a family with $\alpha(\mathcal{F}) \geq t$, $\mathcal{F}^{t,+} \neq \emptyset$ and $\mathcal{F}^{t,-} \neq \emptyset$. Suppose $2 \leq k < \kappa(\mathcal{F}, t)$ and $\mathcal{A}_1, ..., \mathcal{A}_k$ are cross-t-intersecting subfamilies of $\mathcal{F}$ such that $\sum_{i=1}^{k} |\mathcal{A}_i|$ is maximum. Then we neither have $\mathcal{A}_i = \mathcal{F}$ for some $i \in [k]$ and $\mathcal{A}_j = \emptyset$ for all $j \in [k]\backslash\{i\}$ nor have $\mathcal{A}_1 = ... = \mathcal{A}_k = \mathcal{L}$ for some largest t-intersecting subfamily $\mathcal{L}$ of $\mathcal{F}$.*

Proof. By Theorem 1.2, we do not have $\mathcal{A}_1 = ... = \mathcal{A}_k = \mathcal{L}$. Let $\mathcal{B}_1 = \mathcal{F}$ and $\mathcal{B}_2 = ... = \mathcal{B}_k = \mathcal{F}^{t,+}$. Since $\mathcal{B}_1, ..., \mathcal{B}_k$ are cross-t-intersecting, $\sum_{i=1}^{k} |\mathcal{A}_i| \geq \sum_{i=1}^{k} |\mathcal{B}_i|$. So $\sum_{i=1}^{k} |\mathcal{A}_i| \geq |\mathcal{F}| + (k-1)|\mathcal{F}^{t,+}| > |\mathcal{F}|$. The result follows. □

Note that if $\mathcal{F}$ is as in the above proposition, then for all $k \geq 2$, the trivial configuration is not optimal; this is immediate from the proof of the proposition.

An example of a family $\mathcal{F}$ as in the above result is the one in Example 3.4. The example below shows that the phenomenon described at the end of Remark 4.4 may also happen when $\mathcal{F}^{t,+} = \emptyset$ and hence $\mathcal{F} = \mathcal{F}^{t,-}$ (it cannot happen when $\mathcal{F}^{t,-} = \emptyset$, because then $\mathcal{F}$ itself is t-intersecting and hence we can take $\mathcal{A}_1 = ... = \mathcal{A}_k = \mathcal{F}$).

Example 4.6 Let $2 \leq m < k < n$. Let $\mathcal{F} = \{F_1, ..., F_{n+m}\}$ be as in Example 3.5. Let $\mathcal{A}_1 = \{F_1, ..., F_{n+1}\}$ and $\mathcal{A}_2 = ... = \mathcal{A}_k = \{F_{n+1}\}$. Then $\mathcal{A}_1, ..., \mathcal{A}_k$ are cross-t-intersecting and $\sum_{i=1}^{k} |\mathcal{A}_i| = n + k > \max\{n + m, 2k\} = \max\{|\mathcal{F}|, k(l(\mathcal{F}, t))\}$.

However, unlike Proposition 4.5, if $\mathcal{F} = \mathcal{F}^{t,-}$, $\beta(\mathcal{F}, t) \neq \frac{l(\mathcal{F},t)}{|\mathcal{F}|}$ and $k < \kappa(\mathcal{F}, t)$, then the trivial configuration may still give a maximum sum.

Example 4.7 Let $\mathcal{F} = \{F_1, ..., F_{n+m}\}$ be as in Example 3.5, and let $2 \leq k \leq m$. We have $\mathcal{F} = \mathcal{F}^{t,-}$. If $\mathcal{A} \subseteq \mathcal{F}$ and $\mathcal{A}^{t,+} \neq \emptyset$, then one of the following holds: (i) $|\mathcal{A}| = 1$, (ii) $\mathcal{A} = \mathcal{A}^{t,+} = \{F_i, F_{n+1}\}$ for some $i \in [n]$, (iii) $\mathcal{A}^{t,+} = \{F_{n+1}\}$ and $\mathcal{A}^{t,-} \subsetneq \{F_1, ..., F_n\}$. It is therefore easy to check that $\beta(\mathcal{F}, t) = \beta(\mathcal{F}, t, \{F_1, ..., F_{n+1}\}) = \frac{1}{n} < \frac{2}{n+m} = \frac{l(\mathcal{F},t)}{|\mathcal{F}|}$. Since $k \leq m < n$, $k < \kappa(\mathcal{F}, t)$. It is also easy to check that if $\mathcal{A} \subseteq \mathcal{F}$, then $|\mathcal{A}^{t,+}| + \frac{1}{k}|\mathcal{A}^{t,-}|$ is maximum if $\mathcal{A} = \mathcal{F}$. By Proposition 4.3, the trivial configuration gives a maximum sum.

4.2 Solutions for various important families

Section 3.2 gives the values of $\beta(\mathcal{F}, t)$ that we know for the families $2^{[n]}$, $\binom{[n]}{r}$, $\mathcal{P}_{r,n}$, $\mathcal{P}_n^{(r)}$ and $\mathcal{S}_{n,r,m}$, and they all turn out to be the maximum possible value $\frac{l(\mathcal{F},t)}{|\mathcal{F}|}$. Thus, by Theorem 4.1, for all these cases we know the maximum sum of sizes of any $k \geq 2$ cross-t-intersecting subfamilies of $\mathcal{F}$, and we also know that at least one of the trivial configuration (see Remark 4.2) and the configuration $\mathcal{A}_1 = ... = \mathcal{A}_k = \mathcal{L}$ is optimal, with the latter being the unique optimal configuration when $k > \frac{|\mathcal{F}|}{l(\mathcal{F},t)}$. We point out that [29, Theorem 2.5] tells us that in addition to this, for the cases we are discussing except for the one with $\mathcal{F} = \mathcal{P}_{3,3}$ and $t = 1$ (see [6]), when $k < \frac{|\mathcal{F}|}{l(\mathcal{F},t)}$ the trivial configuration is the unique optimal configuration if we simply insist that $\mathcal{A}_1 \neq \emptyset$ (for $2^{[n]}$, this emerges from the correspondence with $\mathcal{S}_{n,n,2}$ used in the proof of Theorem 3.10). As pointed out in

Remark 4.2, for the case $k = \frac{|\mathcal{F}|}{l(\mathcal{F},t)}$ there may be other optimal configurations apart from the two mentioned above.

Now recall the family $\mathcal{V}_{n,r}(q)$ defined in Secion 3.2. If $A, B \in \mathcal{V}_{n,r}(q)$ such that $\dim(A \cap B) \geq t$, then, with slight abuse of terminology, we say that A *t-intersects* B. For $\mathcal{V}_{n,r}(q)$, we work with this definition of t-intersection instead of the usual one, and so we define t-intersecting subfamilies, cross-t-intersecting subfamilies, $l(\mathcal{V}_{n,r}(q),t)$, $\beta(\mathcal{V}_{n,r}(q),t)$, and so on, accordingly. The value of $l(\mathcal{V}_{n,r}(q),t)$ was determined in [21]. $\mathcal{V}_{n,r}(q)$ is t-symmetric; see [29, Example 1.3]. By [29, Corollary 2.4], the statement of Theorem 3.8 holds for $\mathcal{V}_{n,r}(q)$. Thus, by applying the argument in the proof of Proposition 3.2 to $\mathcal{V}_{n,r}(q)$, we obtain $\beta(\mathcal{V}_{n,r}(q),t) = \frac{l(\mathcal{V}_{n,r}(q),t)}{|\mathcal{V}_{n,r}(q)|}$. [29, Theorem 2.5] solved the problem of maximising the sum of sizes of $k \geq 2$ cross-t-intersecting subfamilies of $\mathcal{V}_{n,r}(q)$.

5 The maximum product

In this section we restrict our attention to the the problem of maximising the product of sizes of cross-t-intersecting subfamilies of a given family $\mathcal{F}$. Similarly to the two preceding sections, we first reveal some interesting facts and then we provide solutions for various important families.

5.1 General results and observations

Theorem 1.1 tells us that the configuration $\mathcal{A}_1 = ... = \mathcal{A}_k = \mathcal{L}$ (where $\mathcal{L}$ is a largest t-intersecting subfamily of $\mathcal{F}$) gives both a maximum sum and a maximum product of sizes when $k \geq \kappa(\mathcal{F},t)$, and Theorem 1.2 tells us that this configuration never gives a maximum sum when $k < \kappa(\mathcal{F},t)$. However, this configuration may still give a maximum product when $k < \kappa(\mathcal{F},t)$. For example, the main result in [26] tells us that the product of sizes of 2 cross-1-intersecting subfamilies $\mathcal{A}_1$ and $\mathcal{A}_2$ of $\binom{[n]}{r}$ is maximum if $\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{L}$ for some largest 1-intersecting subfamily $\mathcal{L}$ of $\binom{[n]}{r}$, where $\mathcal{L} = \binom{[n]}{r}$ if $n/2 < r \leq n$, and by the classical result in [17], $\mathcal{L}$ is of size $\binom{n-1}{r-1}$ (the size of the 1-intersecting subfamily $\{A \in \binom{[n]}{r} \colon 1 \in A\}$ of $\binom{[n]}{r}$) if $r \leq n/2$; note that if $r < n/2$, then, since $\beta\left(\binom{[n]}{r},1\right) = \frac{|\mathcal{L}|}{\binom{n}{r}} = \frac{r}{n}$ (see Section 3.2), we have $2 < \kappa\left(\binom{[n]}{r},1\right)$. The general cross-$t$-intersection version (also for 2 subfamilies) is given in [28] for n sufficiently large; see Theorem 5.7. Other results of this kind are given in the next subsection. The following tells us that such product results generalise to k subfamilies for any $k \geq 2$.

Lemma 5.1 *Let $\mathcal{L}$ be a largest t-intersecting subfamily of a family $\mathcal{F}$. Suppose that the product of sizes of p cross-t-intersecting subfamilies $\mathcal{B}_1, ..., \mathcal{B}_p$ of $\mathcal{F}$ is maximum if $\mathcal{B}_1 = ... = \mathcal{B}_p = \mathcal{L}$. Then for any $k \geq p$, the product of sizes of k cross-t-intersecting subfamilies $\mathcal{A}_1, ..., \mathcal{A}_k$ of $\mathcal{F}$ is maximum if $\mathcal{A}_1 = ... = \mathcal{A}_k = \mathcal{L}$.*

We first prove the following result, which immediately yields the above result.

Lemma 5.2 *Let $k \geq p$, and let $x_1, ..., x_k, y_1, ..., y_k$ be non-negative real numbers such that $\prod_{i \in I} x_i \leq \prod_{i \in I} y_i$ for any subset I of $[k]$ of size p. Then $\prod_{i=1}^{k} x_i \leq \prod_{i=1}^{k} y_i$.*

Proof. Let mod* represent the usual *modulo operation* with the exception that for any two positive integers a and b, $ba \bmod^* a$ is a instead of 0. We have

$$\left(\prod_{i=1}^{k} x_i\right)^p = \prod_{i=0}^{k-1} \prod_{j=1}^{p} x_{(ip+j) \bmod^* k} \leq \prod_{i=0}^{k-1} \prod_{j=1}^{p} y_{(ip+j) \bmod^* k} = \left(\prod_{i=1}^{k} y_i\right)^p.$$

Hence the result. $\square$

Proof of Lemma 5.1. By our assumption, $\prod_{i\in I}|\mathcal{A}_i| \le (l(\mathcal{F},t))^p$ for any subset I of $[k]$ of size p. By Lemma 5.2 with $x_i = |\mathcal{A}_i|$ and $y_i = l(\mathcal{F},t)$ for all $i \in [k]$, $\prod_{j=1}^{k}|\mathcal{A}_j| \le (l(\mathcal{F},t))^k$. The result follows. $\square$

We now prove Remark 1.4. More precisely, we will show that for any $t \ge 1$ and any $p \ge 3$, there are families $\mathcal{F}$ with $\kappa(\mathcal{F},t) = p$ such that, unlike the case when $k \ge \kappa(\mathcal{F},t)$ (see Theorem 1.1), for any $2 \le k < \kappa(\mathcal{F},t)$, the product of sizes of k cross-t-intersecting subfamilies $\mathcal{A}_1, ..., \mathcal{A}_k$ of $\mathcal{F}$ is not maximum if $\mathcal{A}_1 = ... = \mathcal{A}_k = \mathcal{L}$ for some largest t-intersecting subfamily $\mathcal{L}$ of $\mathcal{F}$. Our aim is to construct a family $\mathcal{P}$ of size p^2 that can be partitioned into p cross-1-intersecting families $\mathcal{P}_1, ..., \mathcal{P}_p$, each of size p, such that for any $i \in [p]$, the p sets $A_{i,1}, ..., A_{i,p}$ in $\mathcal{P}_i$ are disjoint. Then we take $\mathcal{B}$ to be the family obtained from $\mathcal{P}$ by replacing each element u of the union of all sets in $\mathcal{P}$ by t new elements $u_1, ..., u_t$.

Construction 5.3 Let $p \ge 3$ be an integer. Let $m_1, ..., m_p$ and $c_1, ..., c_p$ be distinct real numbers. For any $i, j \in [p]$, let $L_{i,j}$ be the straight line in $\mathbb{R}^2$ obtained from the function $y\colon \mathbb{R} \to \mathbb{R}$ defined by $y(x) = m_i x + c_j$. For any $i, j \in [p]$, let $A_{i,j}$ be the set of all points (i.e. co-ordinates) of intersection of $L_{i,j}$ with the other lines $L_{i',j'}$, i.e.

$$A_{i,j} = \left\{(a,b) \in \mathbb{R}^2 \colon \exists\, i', j' \in [p],\ (i',j') \neq (i,j), \text{ such that } L_{i,j} \text{ intersects } L_{i',j'} \text{ at } (a,b)\right\}.$$

Let $(a_1,b_1), ..., (a_s,b_s)$ be the distinct co-ordinates in the set $\bigcup_{i=1}^{p}\bigcup_{j=1}^{p} A_{i,j}$ of all points of pairwise intersection of these lines, and let $T_{(a_1,b_1)}, ..., T_{(a_s,b_s)}$ be disjoint sets of size t. For any $i, j \in [p]$, let $B_{i,j} = \bigcup_{(a,b)\in A_{i,j}} T_{(a,b)}$; so $B_{i,j}$ is simply the set obtained by replacing each point (a,b) in $A_{i,j}$ by the t elements of the corresponding set $T_{(a,b)}$. For each $i \in [p]$, let $\mathcal{B}_i = \{B_{i,1}, ..., B_{i,p}\}$. Now let $\mathcal{B} = \bigcup_{i=1}^{p}\mathcal{B}_i = \{B_{i,j}\colon i, j \in [p]\}$.

Theorem 5.4 *Let $\mathcal{B}$ be as in Construction 5.3. Let $\mathcal{L}$ be a largest t-intersecting subfamily of $\mathcal{B}$, and let $\mathcal{A}_1, ..., \mathcal{A}_k$ be cross-t-intersecting subfamilies of $\mathcal{B}$. Then:*
(i) $\kappa(\mathcal{B},t) = |\mathcal{L}| = p$;
(ii) if $k \ge \kappa(\mathcal{B},t)$ and $\mathcal{A}_1 = ... = \mathcal{A}_k = \mathcal{L}$, then $\prod_{i=1}^{k}|\mathcal{A}_i|$ is maximum;
(iii) if $k < \kappa(\mathcal{B},t)$ and $\mathcal{A}_1 = ... = \mathcal{A}_k = \mathcal{L}$, then $\prod_{i=1}^{k}|\mathcal{A}_i|$ is not maximum.

Proof. Let $\mathcal{I}$ be a t-intersecting subfamily of $\mathcal{B}$. For each $i \in [p]$, the lines $L_{i,1}, ..., L_{i,p}$ have the same gradient m_i, and hence, since $c_1, ..., c_p$ are distinct, $L_{i,1}, ..., L_{i,p}$ are distinct parallel lines, meaning that no two intersect. Thus, for each $i \in [p]$, $\mathcal{I}$ has at most one of the sets in $\mathcal{B}_i$. So $|\mathcal{I}| \le p$. Now for any $i, i', j, j' \in [p]$ with $i \neq i'$, $L_{i,j}$ intersects $L_{i',j'}$ (at one point) since $m_i \neq m_{i'}$. So $\{A_{1,1}, A_{2,1}, ..., A_{p,1}\}$ is a 1-intersecting family (in fact, $(0,c_1) \in A_{i,1}$ for each $i \in [p]$) of size p, meaning that $\{B_{1,1}, B_{2,1}, ..., B_{p,1}\}$ is a t-intersecting subfamily of $\mathcal{B}$ of size p, and hence a largest t-intersecting subfamily of $\mathcal{B}$. So $|\mathcal{L}| = p = l(\mathcal{B},t)$.

Let $\mathcal{A}$ be a subfamily of $\mathcal{B}$. If $\mathcal{A}^{t,-} = \emptyset$, then $\beta(\mathcal{B},t,\mathcal{A}) = \frac{l(\mathcal{B},t)}{|\mathcal{B}|}$. Suppose $\mathcal{A}^{t,-} \neq \emptyset$. By the same argument for $\mathcal{I}$ above, for each $i \in [p]$, $\mathcal{A}^{t,+}$ has at most one of the sets in $\mathcal{B}_i$, and if it does have one of these sets, then, by definition of $\mathcal{A}^{t,+}$, $\mathcal{A}$ has no other set in $\mathcal{B}_i$. Let $S = \{i \in [p]\colon \mathcal{A}^{t,+}$ has one of the sets in $\mathcal{B}_i\}$. Then $|\mathcal{A}^{t,+}| = |S|$, and for each $s \in S$, $\mathcal{A}^{t,-}$ has no set in $\mathcal{B}_s$. So $\mathcal{A}^{t,-} \subseteq \bigcup_{j\in[p]\setminus S}\mathcal{B}_j$ and hence $|\mathcal{A}^{t,-}| \le (p-|S|)p$. Note that $|S| < p$ since $|\mathcal{A}^{t,-}| > 0$. So we have

$$\beta(\mathcal{B},t,\mathcal{A}) = \frac{l(\mathcal{B},t) - |\mathcal{A}^{t,+}|}{|\mathcal{A}^{t,-}|} \ge \frac{p-|S|}{(p-|S|)p} = \frac{1}{p} = \frac{p}{p^2} = \frac{l(\mathcal{B},t)}{|\mathcal{B}|} = \beta(\mathcal{B},t,\mathcal{L}).$$

Therefore, $\beta(\mathcal{B}, t) = \frac{1}{p}$ and hence $\kappa(\mathcal{B}, t) = p$. Hence (i).

Part (ii) is given by Theorem 1.1.

Finally, suppose $k < \kappa(\mathcal{B}, t)$. So $p \geq k + 1$. Let $\mathcal{B}'_k = \bigcup_{i=k}^{p} \mathcal{B}_i$. As we mentioned above, for $i \neq i'$, any two lines $L_{i,j}$ and $L_{i',j'}$ intersect at a point (a, b), and hence the t-set $T_{(a,b)}$ is a subset of $B_{i,j} \cap B_{i',j'}$. So $\mathcal{B}_1, ..., \mathcal{B}_{k-1}, \mathcal{B}'_k$ are cross-t-intersecting subfamilies of $\mathcal{B}$, and the product of their sizes is $p^{k-1}(p - k + 1)p > p^k = |\mathcal{L}|^k$. Hence (iii). □

5.2 Solutions for various important families

The cross-t-intersection problem for the product is more difficult than that for the sum, and hence less is known about the product. However, various breakthroughs have been made for the special families in Section 3.2.

Consider first the family $2^{[n]}$. For $t = 1$ we have the complete solution given by Theorem 3.7, and for $t \geq 1$ we have the following.

Theorem 5.5 ([25]) *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be cross-t-intersecting subfamilies of $2^{[n]}$, where $1 \leq t \leq n$. Let $\mathcal{K}_1 = \{A \subseteq [n] \colon |A| \geq (n+t)/2\}$, $\mathcal{K}_2 = \{A \subseteq [n] \colon |A \cap [n-1]| \geq (n+t-1)/2\}$ and $\mathcal{K}_3 = \{A \subseteq [n] \colon |A| \geq (n+t-1)/2\}$.*
(i) If $n + t$ is even, then $|\mathcal{A}_1||\mathcal{A}_2| \leq |\mathcal{K}_1|^2$.
(ii) If $n + t$ is odd, then $|\mathcal{A}_1||\mathcal{A}_2| \leq \max\{|\mathcal{K}_2|^2, |\mathcal{K}_1||\mathcal{K}_3|\}$.

Thus, by Lemma 5.1, if $n + t$ is even, then the product of $k \geq 2$ cross-t-intersecting subfamilies $\mathcal{A}_1, ..., \mathcal{A}_k$ of $2^{[n]}$ is maximum if $\mathcal{A}_1 = ... = \mathcal{A}_k = \mathcal{K}_1$. Lemma 5.1 yields a similar generalisation for the case when $n + t$ is odd and $|\mathcal{K}_2|^2 \geq |\mathcal{K}_1||\mathcal{K}_3|$. However, it is not known what the maximum product is when $n + t$ is odd, $k \geq 3$ and $|\mathcal{K}_2|^2 < |\mathcal{K}_1||\mathcal{K}_3|$, and we conjecture that it is $\max\{|\mathcal{K}_2|^k, |\mathcal{K}_1|^{k-1}|\mathcal{K}_3|\}$.

The following theorems were proved for 2 subfamilies, and for each one of them, we obtain the generalisation to any $k \geq 2$ subfamilies from Lemma 5.1 (with $p = 2$).

For the family $\binom{[n]}{r}$, we have the next two results.

Theorem 5.6 ([27, 26]) *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be cross-1-intersecting subfamilies of $\binom{[n]}{r}$, where $1 \leq r \leq n/2$. Then*

$$|\mathcal{A}_1||\mathcal{A}_2| \leq \binom{n-1}{r-1}^2,$$

and equality holds if $\mathcal{A}_1 = \mathcal{A}_2 = \left\{A \in \binom{[n]}{r} \colon 1 \in A\right\}$.

The result for $n/2 < r \leq n$ is trivial; in this case, $\mathcal{A}_1$ and $\mathcal{A}_2$ are cross-1-intersecting if each one of them is the whole family $\binom{[n]}{r}$.

Theorem 5.7 ([28]) *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be cross-t-intersecting subfamilies of $\binom{[n]}{r}$, where $1 \leq t \leq r$. If n is sufficiently large, then*

$$|\mathcal{A}_1||\mathcal{A}_2| \leq \binom{n-t}{r-t}^2,$$

and equality holds if $\mathcal{A}_1 = \mathcal{A}_2 = \left\{A \in \binom{[n]}{r} \colon [t] \subset A\right\}$.

Finally, for $\mathcal{P}_{n,n}$ we have the next two results.

Theorem 5.8 ([15]) *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be cross-1-intersecting subfamilies of $\mathcal{P}_{n,n}$, where $n \geq 4$. Then*

$$|\mathcal{A}_1||\mathcal{A}_2| \leq ((n-1)!)^2,$$

and equality holds if $\mathcal{A}_1 = \mathcal{A}_2 = \{A \in \mathcal{P}_{n,n} \colon (1,1) \in A\}$.

Theorem 5.9 ([16]) *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be cross-t-intersecting subfamilies of $\mathcal{P}_{n,n}$. If n is sufficiently large, then*

$$|\mathcal{A}_1||\mathcal{A}_2| \leq ((n-t)!)^2,$$

and equality holds if $\mathcal{A}_1 = \mathcal{A}_2 = \{A \in \mathcal{P}_{n,n} \colon \{(1,1), ..., (t,t)\} \subset A\}$.

Acknowledgement. The author is indebted to the anonymous referee for checking the paper carefully and providing remarks that led to an improvement in the presentation.

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